

# INDIFFERENCE PRICING AND THE MINIMAL ENTROPY MARTINGALE MEASURE IN A STOCHASTIC VOLATILITY MODEL WITH JUMPS

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**ABSTRACT.** We use the dynamic programming approach to derive an equation for the utility indifference price of Markovian claims in a stochastic volatility model proposed by Barndorff-Nielsen and Shephard [3]. The pricing equation is a Black & Scholes equation with a nonlinear integral term involving the risk preferences of the investor. Passing to the zero risk aversion limit, we present a Feynman-Kac representation of the minimal entropy price. The density of the minimal entropy martingale measure is found via the Girsanov transform of the Brownian motion and a subordinator process controlling the jumps in the volatility model. The density is represented by the logarithm of the value function for an investor with exponential utility and no claim issued, and a Feynman-Kac representation of this function is provided. We calculate the function explicitly in a special case, and show some properties in the general case.

## 1. INTRODUCTION

Utility indifference pricing (see Hodges and Neuberger [9]) gives an alternative to the arbitrage theory to derive the fair premium of derivatives in incomplete markets. It is well-known that in such markets there exists a continuum of equivalent martingale measures, and the arbitrage theory does not lead in general to a unique price. Hence, the investors attitude towards the risk that can not be hedged away must be taken into account in the problem of pricing derivatives in incomplete markets.

In this paper we will study the problem of pricing Markovian claims in a stochastic volatility model introduced by Barndorff-Nielsen and Shephard [3]. The price dynamics of the underlying follows a geometric Brownian motion where the squared volatility is modelled by a non-Gaussian Ornstein-Uhlenbeck process. The volatility level will revert towards zero, with random upward shifts modelled by a subordinator process (an increasing Lévy process).

Following Hodges and Neuberger [9] we consider an investor trying to maximize his exponential utility by either entering into the market by his own account, or issuing a derivative and investing his incremental wealth after collecting the premium. The indifference price of

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the claim is then defined as the premium for which the investor becomes indifferent between the two investment alternatives. In this paper we use the dynamic programming approach to solve the two utility maximization problems. Since the volatility process follows a jump diffusion model, we obtain Hamilton-Jacobi-Bellman (HJB) equations with integral terms. When the investor enters the market without issuing a claim, we solve this problem via a logarithmic transform of the value function and a Feynman-Kac representation of the transform. This function is of crucial importance when considering the portfolio problem with a short position in the derivative, and we thus analyze the function in some detail and provide explicit solution in a special case.

We continue with deriving the corresponding HJB-equation when the investor optimizes his portfolio with an issued derivative. Again we can represent the solution via a logarithmic transform, however, now this transform includes the indifference pricing functions for which we are able to derive a Black & Scholes type of partial differential equation with a nonlinear integral term depending on the risk preferences. Unfortunately, we are not able to present any solution of this equation. We remark that our solution approach to these stochastic control problems follows the same lines as in Musiela and Zariphopoulou [14], who consider indifference pricing for claims written on non-tradeable assets. In their framework of continuous diffusion processes, they are able to derive explicit solutions also for the indifference price via a power transformation of the nonlinear pricing equation.

It is well-known (see e.g. Frittelli [11], Rouge and El-Karoui [17] and Delbaen *et al.* [7]) that the zero risk aversion limit of the indifference price corresponds to the minimal entropy martingale measure price. After formally taking the limit in our Black & Scholes integro-equation for the indifference price, we obtain a *linear* Black & Scholes integral equation, for which we present a Feynman-Kac solution. Reading off the corresponding Girsanov transform, we obtain a candidate density for the minimal entropy martingale measure. We verify that this is indeed the minimal entropy martingale measure by appealing to a verification theorem derived by Rheinländer [16]. A crucial ingredient in this analysis is the logarithmic transform of the value function when no claim is issued. Related papers studying the minimal entropy martingale measure for stochastic volatility markets are Hobson [8], Becherer [4] and Benth and Karlsen [5].

The paper is organized as follows: in the next section we define our financial market, and in Section 3 the different optimization problems are presented and analysed. The next Section identifies the candidate for the minimal entropy martingale measure and the entropy price, while in Section 5 we verify under some integrability conditions that this is the desired measure.

## 2. THE MARKET

Given a probability space  $(\Omega, \mathcal{F}, P)$  and a time horizon  $T$ , consider a financial market consisting of a bond and a risky asset with prices at time  $0 \leq t \leq T$  denoted by  $R_t$  and  $S_t$ , resp.. Assume without loss of generality that the bond yields a risk-free rate of return equal to zero, i.e.,

$$(2.1) \quad dR_t = 0,$$

together with the convention that  $R_0 = 1$ . The price of the risky asset is evolving according to the following stochastic volatility model introduced by Barndorff-Nielsen & Shepard [3];

$$(2.2) \quad dS_t = \alpha(Y_t)S_t dt + \sigma(Y_t)S_t dB_t, \quad S_0 = s > 0$$

$$(2.3) \quad dY_t = -\lambda Y_t dt + dL_{\lambda t}, \quad Y_0 = y > 0,$$

where  $B_t$  is a Brownian motion and  $L_t$  a subordinator (that is, an increasing Lévy process) with Poisson random measure denoted by  $N(dt, dz)$ . The Lévy measure  $\nu(dz)$  of  $L_t$  satisfies  $\int_0^\infty \min(1, z) \nu(dz) < \infty$ . Further, we denote by  $\{\mathcal{F}_t\}_{t \geq 0}$  the completion of the filtration  $\sigma(B_s, L_{\lambda s}; s \leq t)$  generated by the Brownian motion and the subordinator such that  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  becomes a complete filtered probability space. In this paper we will consider the following specification of the parameter functions  $\alpha$  and  $\sigma$ :

$$(2.4) \quad \alpha(y) = \mu + \beta y, \quad \sigma(y) = \sqrt{y},$$

with  $\mu$  and  $\beta$  being constants.

The process  $Y_t$  models the squared volatility, and will be an Ornstein-Uhlenbeck process reverting towards zero, and having positive jumps given by the subordinator. An explicit representation of the squared volatility is

$$(2.5) \quad Y_t = y \exp(-\lambda t) + \int_0^t \exp(-\lambda(t-u)) dL_{\lambda u}.$$

The scaling of time by  $\lambda$  in the subordinator is to decouple the modelling of the marginal distribution of the (log)returns of  $S$  and their autocorrelation structure. We note that [3] propose the use a superposition of processes  $Y_t$  with different speed of mean-reversion. However, in this paper we will stick to only one process  $Y_t$ , but remark that there are no essential difficulties in generalizing to the case of a superposition of  $Y$ 's. The modelling idea is to specify a stationary distribution of  $Y$  that implies (at least approximately) a desirable distribution for the returns of  $S$ . Given this stationary distribution, one needs to derive a subordinator  $L$ . In [3] several examples of such distributions and their associated subordinators are given in the context of financial applications.

We denote by  $\psi(\theta)$  the cumulant function of  $L_t$ , which is defined as the logarithm of the characteristic function

$$(2.6) \quad \psi(\theta) = \ln \mathbb{E} [\exp(i\theta L_1)], \quad \theta \in \mathbb{R}.$$

From the Lévy-Kintchine Formula we have

$$(2.7) \quad \psi(\theta) = \int_0^\infty \{e^{i\theta z} - 1\} \nu(dz).$$

We suppose that the Lévy measure satisfies an exponential integrability condition, that is, there exists a constant  $c > 0$  such that

$$(2.8) \quad \int_1^\infty e^{cz} \nu(dz) < \infty.$$

Later we will be more precise about the size of  $c$ , and relate it to parameters in the specification of the Lévy measure. Under this condition, the moment generating function

is defined for all  $|\theta| \leq c$ , and

$$(2.9) \quad \mathbb{E}[\exp(\theta L_1)] = \exp(\phi(\theta))$$

where

$$(2.10) \quad \phi(\theta) = \int_0^\infty \{e^{\theta z} - 1\} \nu(dz).$$

Note that  $L_{\lambda t}$  is also a subordinator, and the cumulant function of this is  $\lambda\psi(\theta)$ . The process  $L_t$  has the decomposition

$$(2.11) \quad L_t = \int_0^t \int_0^\infty z \nu(dz) dt + \int_0^t \int_0^\infty z (N(dz, dt) - \nu(dz) dt),$$

where the second integral on the right-hand side is a martingale. The reader is referred to [1], [6], [15] and [19] for more information about Lévy processes and subordinators.

### 3. INDIFFERENCE PRICING OF CLAIMS

In this Section we will use the dynamic programming approach to determine the density of the minimal entropy martingale measure. By considering the utility maximization problems for the issuer of a claim, we are able to associate an integro-partial differential equation for the indifference price. By letting the risk aversion of the investor tend to zero, we formally obtain a limiting equation, being a Black & Scholes type equation for which we can associate a Feynman-Kac solution. From this representation, we can read off the density of the minimal entropy martingale measure. A basic ingredient in the density is a function factorizing the solution of the optimization when no claim is issued. We characterize this function, and provide an explicit form of it in a special (but interesting) case.

**3.1. The exponential utility optimization problems.** Consider a European option with a Markovian claim defined by  $f(S_T)$ , for a bounded function  $f$ . Let the investor have an exponential utility function

$$U(x) = 1 - \exp(-\gamma x),$$

where  $\gamma > 0$  is the risk aversion parameter. The investor, being an agent in the market (2.1)-(2.2) with initial wealth  $x$  at time  $t$ , has a wealth dynamics  $X_u$  at time  $u \geq t$  governed by the equation

$$(3.1) \quad dX_u = \pi_u \alpha(Y_u) X_u du + \pi_u \sigma(Y_u) X_u dB_u, \quad X_t = x,$$

where  $\pi_u$  denotes the fraction of the wealth  $X_u$  which is invested in the risky asset  $S_u$  at time  $u$ . The control  $\pi$  is called *admissible* if it is an  $\mathcal{F}_u$ -adapted stochastic process for which there exists a wealth process  $X_u^\pi$  solving the stochastic differential equation (3.1). We denote the set of all such controls by  $\mathcal{A}_t$ , where the subscript  $t$  indicates that we start the wealth dynamics at time  $t$ .

Note that the admissible controls depend on the level of volatility  $Y$ , and not only on the stock price which are directly observable. However, this is not any restriction in the current stochastic volatility model, where in fact the investor has full knowledge of the volatility

from observing the stock price and its quadratic variation due to the positivity of the volatility. Indeed, by taking the quadratic variation  $[S]_t$  of  $S_t$  and solving for  $\sigma(Y_t) = Y_t$ :

$$Y_t = S_t^{-2} \frac{d[S]_t}{dt},$$

we find after appealing to the fact  $\sigma(Y_s; s \leq t) = \sigma(L_{\lambda s}; s \leq t)$ , that the filtration  $\mathcal{F}_t$  equals  $\mathcal{F}_t^S$ , the filtration generated by the asset price  $S$ .

Restricting our attention to Markov controls, the investor will allocate a fraction  $\pi \equiv \pi(t, x, y)$  into the risky asset when the wealth is  $X_t = x$  and level of volatility is  $Y_t = y$ . The value function for the optimal control problem, given that the investor has *not* issued a claim, is

$$(3.2) \quad V^0(t, x, y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E} [1 - \exp(-\gamma X_T) | X_t = x, Y_t = y].$$

If, on the other hand, the investor issues a claim  $f(S_T)$ , the utility maximization problem is

$$(3.3) \quad V(t, x, s, y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E} [1 - \exp(-\gamma(X_T - g(S_T))) | X_t = x, Y_t = y, S_t = s].$$

Following Hodges and Neuberger [9], the *utility indifference price* of the claim  $f(S_T)$  for a given risk aversion  $\gamma$ , is now defined as the unique solution  $\Lambda^{(\gamma)}(t, y, s)$  of the equation

$$(3.4) \quad V^0(t, x, y) = V(t, x + \Lambda^{(\gamma)}(t, y, s), s, y).$$

The purpose of the rest of this section is to solve the two utility optimization problems and reach an integro-partial differential equation for the price  $\Lambda^{(\gamma)}$ .

We shall use the dynamic programming (or Bellman) method to solve the two stochastic control problems. Provided that the value functions are sufficiently regular, it is well known that the associated Hamilton-Jacobi-Bellman (HJB henceforth) equations can be derived using the dynamic programming principle.

**3.2. Utility optimization without a claim issued.** The HJB equation for the value function (3.2) without a claim issued reads

$$(3.5) \quad \begin{aligned} V_t^0 + \max_{\pi \in \mathbb{R}} \left\{ \alpha(y) \pi x V_x^0 + \frac{1}{2} \sigma^2(y) \pi^2 x^2 V_{xx}^0 \right\} \\ + \mathcal{L}_Y V^0 = 0, \quad (t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+, \end{aligned}$$

with terminal data

$$(3.6) \quad V^0(T, x, y) = 1 - \exp(-\gamma x), \quad (x, y) \in \mathbb{R} \times \mathbb{R},$$

where

$$(3.7) \quad \mathcal{L}_Y V^0 = -\lambda y V_y^0 + \lambda \int_0^\infty \{V^0(t, x, y + z) - V^0(t, x, y)\} \nu(dz).$$

The first order condition for an optimal investment strategy is

$$\alpha(y) x V_x^0 + \sigma^2(y) \pi x^2 V_{xx}^0 = 0,$$

and the solution  $\pi^*$  of this equation is

$$\pi^* = -\frac{\alpha(y)V_x^0}{\sigma^2(y)xV_{xx}^0}.$$

Inserting  $\pi^*$  into the HJB equation (3.5) yields the nonlinear integro-PDE

$$(3.8) \quad V_t^0 - \frac{\alpha^2(y)(V_x^0)^2}{2\sigma^2(y)V_{xx}^0} + \mathcal{L}_Y V^0 = 0, \quad (t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+.$$

We reduce the state space by one dimension by making the ansatz (see Musiela and Zariphopoulou [14] for a similar ansatz in a different model)

$$(3.9) \quad V^0(t, x, y) = 1 - \exp(-\gamma x)H(t, y).$$

This logarithmic transform simplifies the nonlinearities in (3.8) considerably, and insertion of the ansatz in (3.8) yields the following linear integro-PDE for  $H(t, y)$

$$(3.10) \quad H_t - \frac{\alpha^2(y)}{2\sigma^2(y)}H + \mathcal{L}_Y H = 0, \quad (t, y) \in [0, T) \times \mathbb{R}_+,$$

with terminal data induced by (3.6)

$$(3.11) \quad H(T, y) = 1, \quad y \in \mathbb{R}_+.$$

The function  $H$  solving (3.10)-(3.11) plays a crucial role in the derivation of the density of the minimal entropy martingale measure. We next prove that a smooth solution of the integro-PDE (3.10)-(3.11) exists and present its Feynman-Kac representation:

**Proposition 3.1.** *Equation (3.10)-(3.11) has a solution  $H \in C^{1,1}([0, T] \times \mathbb{R}_+)$  which allows for the following Feynman-Kac representation*

$$(3.12) \quad H(t, y) = \mathbb{E} \left[ \exp \left( -\frac{1}{2} \int_t^T \frac{\alpha^2(Y_u)}{\sigma^2(Y_u)} du \right) \middle| Y_t = y \right], \quad (t, y) \in [0, T] \times \mathbb{R}_+.$$

*Proof.* First, we note that from Markov theory a sufficiently smooth solution of (3.10)-(3.11) will have the Feynman-Kac representation (3.12). We prove that  $H$  is continuously differentiable in  $t$  and  $y$ :

Denote

$$(3.13) \quad g(y) := -\frac{1}{2} \frac{\alpha^2(y)}{\sigma^2(y)} = -\frac{1}{2} \left( \frac{\mu^2}{y} + 2\mu\beta + \beta^2 y \right).$$

Using the explicit solution of  $Y_u$  in (2.5) given  $Y_t = y$  together with  $Y_u \geq y \exp(-\lambda(u-t))$ , we easily see by appealing to dominated convergence that  $H$  is continuously differentiable with respect to  $t$  and  $y$ , and that the differentiation in  $y$  is continuous. Further, for the differentiation in  $t$  we have

$$(3.14) \quad |H_t(t, y) - H_t(\tau, y)| \leq \mathbb{E} \left[ |g(Y_t) \exp \left( \int_t^T g(Y_u) du \right) - g(Y_\tau) \exp \left( \int_\tau^T g(Y_u) du \right)| \right] \\ \leq \mathbb{E} \left[ \exp \left( \int_t^T g(Y_u) du \right) |g(Y_t) - g(Y_\tau)| \right]$$

$$+ \mathbb{E} \left[ \left| \exp \left( \int_{\tau}^t g(Y_u) du \right) - 1 \right| |g(Y_{\tau})| \exp \left( \int_0^{\tau} g(Y_u) du \right) \right]$$

We observe that the second term of the sum on the right hand side in (3.14) goes to zero as  $\tau \rightarrow t$ . For the first term, note that

$$(g(Y_t) - g(Y_{\tau}))^2 \leq 2\mu^4 \left( \frac{1}{Y_t} - \frac{1}{Y_{\tau}} \right)^2 + 2\beta^4 (Y_t - Y_{\tau})^2.$$

Hence, we conclude that  $H$  is continuously differentiable in  $t$  by using

$$\left( \frac{1}{Y_t} - \frac{1}{Y_r} \right)^2 = \frac{(Y_t - Y_r)^2}{Y_r^2 Y_t^2} \leq \frac{1}{y^2} \exp(2\lambda(r - t))(Y_t - Y_r)^2$$

together with Hölder inequality and the fact that  $\mathbb{E}[(Y_t - Y_{\tau})^2]$  can be dominated by  $t - \tau$ .  $\square$

We sum up our findings for the utility optimization problem without a claim issued in the following proposition:

**Proposition 3.2.** *The value function of the utility optimization problem stated in (3.2) is*

$$V^0(t, x, y) = 1 - \exp(-\gamma x) H(t, y),$$

where  $H$  is defined in Prop. 3.12. Furthermore, the optimal investment strategy is the feedback control

$$\pi^*(t, x, y) = \frac{1}{\gamma x} \left( \frac{\mu}{y} + \beta \right).$$

*Proof.* First, we notice that  $V^0$  is a bounded and smooth function. By appealing to standard arguments, one can prove a verification theorem which will identify  $V^0$  as the value function of the control problem and the optimal control being  $\pi^*$ . We refer the reader to, e.g. Fleming and Soner [10].  $\square$

In general, (3.12) is rather difficult to calculate explicitly. However, if we consider the special case  $\alpha(y) = \beta y$ , i.e  $\mu = 0$  in (2.2), a direct calculation using the moment generating function of  $L_1$  gives the following explicit solution of the integro-PDE (3.10)-(3.11):

**Corollary 3.3.** *Suppose  $\alpha(y) = \beta y$ . Then the solution of (3.10)-(3.11) is given as*

$$(3.15) \quad H(t, y) = \exp(b(t)y + c(t)),$$

where  $b$  and  $c$  are defined as

$$b(t) = -\frac{\beta^2}{2\lambda} (1 - \exp(-\lambda(T - t))), \quad c(t) = \lambda \int_t^T \phi(b(u)) du,$$

We recall that  $\phi$  is the log moment generating function of  $L_1$ .

Setting  $\mu = 0$  in (2.2) corresponds to an expected logreturn of  $(\beta - \frac{1}{2})y$  of the risky asset  $S_t$ . If we, for instance, specify the stationary distribution of  $Y$  to be inverse Gaussian, then

the logreturns will be approximately normal inverse Gaussian distributed (see Barndorff-Nielsen and Shephard [3]), and choosing this to be symmetric corresponds to  $\beta = \frac{1}{2}$ , that is, with  $\mu = 0$  we have zero expected logreturn.

One can use the representation in (3.12) of  $H$  to extract a lower bound for the function, which we now derive;

**Proposition 3.4.** *Define*

$$\begin{aligned} a(t) &= -\frac{\mu}{2\lambda} (\exp(\lambda(T-t)) - 1), \\ b(t) &= -\frac{\beta^2}{2\lambda} (1 - \exp(-\lambda(T-t))), \\ c(t) &= -\mu\beta(T-t) + \lambda \int_t^T \phi(b(u)) du. \end{aligned}$$

Then we have the following bounds for  $H(t, y)$

$$(3.16) \quad \exp(a(t)y^{-1} + b(t)y + c(t)) \leq H(t, y) \leq 1.$$

*Proof.* The upper bound of 1 is clear (which is reached for  $t = T$ ). Using (3.13), together with the explicit representation of  $Y_u$  in (2.5), its lower bound  $Y_u \geq y \exp(-\lambda(u-t))$  and the fact that

$$-\lambda \int_t^T Y_u du = Y_T - Y_t - (L_{\lambda T} - L_{\lambda t}),$$

it is straightforward to derive

$$(3.17) \quad H(t, y) \geq \exp(a(t)y^{-1} + b(t)y - \mu\beta(T-t)) \cdot \mathbb{E} \left[ \exp \left( -\frac{\beta^2}{2\lambda} \int_t^T (1 - \exp(-\lambda(T-u))) dL_{\lambda u} \right) \right],$$

which completes the proof.  $\square$

**3.3. Utility optimization with a claim issued.** Next, consider the HJB equation for the value function (3.3) when the investor has issued a claim with payoff function  $g(s)$  at time  $T$ :

$$(3.18) \quad \begin{aligned} V_t + \max_{\pi \in \mathbb{R}} \left\{ \alpha(y)\pi x V_x + \frac{1}{2}\sigma^2(y)\pi^2 x^2 V_{xx} + \sigma^2(y)\pi x s V_{xs} \right\} \\ + \mathcal{L}_S V + \mathcal{L}_Y V = 0, \quad (t, x, y, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \end{aligned}$$

with terminal data

$$(3.19) \quad V(T, x, y, s) = 1 - \exp(-\gamma(x - f(s))),$$

where  $\mathcal{L}_Y$  is defined in (3.7) and

$$\mathcal{L}_S V = \alpha(y)s V_s + \frac{1}{2}\sigma^2(y)s^2 V_{ss}.$$



From the first order condition we can derive the following expression for the optimal investment strategy  $\pi^*$ :

$$(3.20) \quad \pi^* = -\frac{\alpha(y)V_x + \sigma^2(y)sV_{xs}}{\sigma^2(y)xV_x}.$$

Inserting  $\pi^*$  into the HJB equation (3.18) yields the integro-PDE

$$(3.21) \quad V_t - \frac{\alpha^2(y)V_x^2}{2\sigma^2(y)V_{xx}} - \frac{\sigma^2(y)s^2V_{xs}^2}{2V_{xx}} - \frac{\alpha(y)sV_xV_{xs}}{V_{xx}} + \mathcal{L}_S V + \mathcal{L}_Y V = 0, \quad (t, x, y, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+.$$

We now make the ansatz

$$(3.22) \quad V(t, x, y) = 1 - \exp(-\gamma x + \gamma \Lambda^{(\gamma)}(t, y, s)) H(t, y),$$

where we recall that  $\Lambda^{(\gamma)}(t, y, s)$  is the indifference price to be determined and  $H(t, y)$  solving (3.10)-(3.11). We can derive an integro-PDE for  $\Lambda^{(\gamma)}$ : after some simple manipulations, plugging (3.22) into (3.21) and using the equation (3.10) for  $H$ , we derive the following integro-PDE for  $\Lambda^{(\gamma)}$  for  $(t, y, s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$ :

$$(3.23) \quad \Lambda_t^{(\gamma)} + \frac{1}{2}\sigma^2(y)s^2\Lambda_{ss}^{(\gamma)} - \lambda y\Lambda_y^{(\gamma)} + \lambda \int_0^\infty \frac{1}{\gamma} \left\{ \exp(\gamma(\Lambda^{(\gamma)}(t, y+z, s) - \Lambda^{(\gamma)}(t, y, s))) - 1 \right\} \frac{H(t, y+z)}{H(t, y)} \nu(dz) = 0.$$

Also, since (3.19) holds,  $\Lambda^{(\gamma)}$  obeys the terminal condition

$$(3.24) \quad \Lambda^{(\gamma)}(T, y, s) = f(s), \quad (y, s) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Unfortunately, we are not able to provide any solution of (3.23)-(3.24), and therefore we can not verify that  $V$  defined in (3.22) indeed is the value function and  $\pi^*$  defined in (3.20) is the optimal control of our optimization problem. Except for the exponential function in the integral term of (3.23), the indifference price for general risk aversion  $\gamma$  follows a linear Black & Scholes-type partial differential equation.

#### 4. IDENTIFICATION OF THE CANDIDATE MINIMAL ENTROPY MARTINGALE MEASURE AND THE ENTROPIC PRICE

The entropy price of the claim  $f(S_T)$  occurs as the zero risk aversion limit

$$\Lambda(t, y, s) := \lim_{\gamma \rightarrow 0} \Lambda^{(\gamma)}(t, y, s).$$

Taking formally this limit in (3.23), the following integro-PDE for  $\Lambda(t, y, s)$  appears for  $(t, y, s) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}_+$ ,

$$(4.1) \quad \Lambda_t + \frac{1}{2} \sigma^2(y) s^2 \Lambda_{ss} - \lambda y \Lambda_y + \lambda \int_0^\infty (\Lambda(t, y+z, s) - \Lambda(t, y, s)) \frac{H(t, y+z)}{H(t, y)} \nu(dz) = 0.$$

The terminal condition  $\Lambda(T, y, s) = f(s)$  yields the Feynman-Kac representation

$$(4.2) \quad \Lambda(t, y, s) = \mathbb{E} \left[ f(\tilde{S}_T) \mid \tilde{Y}_t = y, \tilde{S}_t = s \right],$$

where the stochastic processes  $\tilde{S}$  and  $\tilde{Y}$  are given by

$$(4.3) \quad d\tilde{S}_t = \sigma(\tilde{Y}_t) \tilde{S}_t d\tilde{B}_t,$$

$$(4.4) \quad d\tilde{Y}_t = -\lambda \tilde{Y}_t dt + d\tilde{L}_{\lambda t},$$

and  $\tilde{L}_t$  is a pure jump Markov process with jump measure

$$(4.5) \quad \tilde{\nu}(\omega, dz, dt) = \frac{H(t, \tilde{Y}_t(\omega) + z)}{H(t, \tilde{Y}_t(\omega))} \nu(dz) dt.$$

Observe that the state-dependent jump measure  $\tilde{\nu}(dz)$  becomes deterministic when  $\mu = 0$ : indeed, from Cor. 3.3 we find that

$$\tilde{\nu}(\omega, dz, dt) = e^{b(t)z} \nu(dz) dt,$$

where  $b(t)$  is given in Cor. 3.3.

Introduce the notation

$$(4.6) \quad \delta(y, z, t) := \frac{H(t, y+z)}{H(t, y)}.$$

Our interest is now to identify (formally) a candidate for a martingale measure  $Q$  such that the representation (4.2) can be rewritten in terms of the original processes  $S_t$  and  $Y_t$ . Since  $B_t$  and  $L_t$  are independent, we proceed in two step. By the Girsanov theorem for Brownian motion, we see that

$$(4.7) \quad Z'_T = \exp \left( - \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t - \int_0^T \frac{1}{2} \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right)$$

is a density candidate to change from the dynamics of  $S_t$  to the dynamics of  $\tilde{S}_t$ . In a second step we look for a probability that causes the dynamic change from  $Y_t$  to  $\tilde{Y}_t$ . Using the Girsanov theorem for random measures (see Jacod and Shiryaev [12]), we get the following density candidate

$$(4.8) \quad Z''_T = \exp \left( \int_0^T \int_0^\infty \ln \delta(Y_t, z, t) N(dz, dt) + \int_0^T \int_0^\infty (1 - \delta(Y_t, z, t)) \nu(dz) dt \right).$$

Now, since  $Z'_T$  and  $Z''_T$  are orthogonal, a natural candidate for the density of the minimal entropy martingale measure is

$$(4.9) \quad Z_T := Z'_T \cdot Z''_T.$$

## 5. VERIFICATION OF THE CANDIDATE MINIMAL ENTROPY MARTINGALE MEASURE

In this section, we want to prove that our candidate  $Z_T$  in (4.9) is indeed the density process of the minimal entropy martingale measure. To this end, we need to verify that  $Z_T$  is a martingale (not only a local martingale) defining a probability measure with finite relative entropy, which moreover is minimal among all probability measures of finite relative entropy. We will do this by verifying the sufficient conditions developed by Rheinländer [16].

The main result in this paper is the following theorem:

**Theorem 5.1.** *Suppose we have*

$$(5.1) \quad \mathbb{E} \left[ \exp \left( \int_0^T \frac{\alpha^2(Y_s)}{\sigma^2(Y_s)} ds \right) \right] < \infty.$$

*Then  $Z_t$  as defined in (4.9) is the minimal entropy martingale measure density process.*

*Proof.* Referring to the results in [16], it is enough to verify the following four statements

**i):** The density candidate  $Z_T$  can be written as

$$(5.2) \quad Z_T = \exp \left( c + \int_0^T \eta_t dS_t \right),$$

for a constant  $c$  and some adapted process  $\eta_t$ .

**ii):** The process  $Z_t$  is a true martingale.

**iii):** The measure induced by  $Z_t$ , denoted by  $Q_{ME}$  has finite entropy.

**iv):** We have

$$\int_0^T \eta_t^2 d[S]_t \in L_{exp}(P),$$

where  $[S]_t$  is the quadratic variation process of  $S_t$  and  $L_{exp}(P)$  is the Orlicz space generated by the Young function  $\exp(\cdot)$ .

**i)** We want to write  $Z_T$  as in (5.2). Since we have

$$\frac{dS_t}{S_t} = \alpha(Y_t)dt + \sigma(Y_t)dB_t,$$

we get

$$(5.3) \quad \ln(Z'_T) = - \int_0^T \frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1} dS_t + \frac{1}{2} \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt.$$

Now, substituting in (5.3) for  $\frac{1}{2} \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}$  the expression we get from the integro-PDE (3.10), we end up with

$$(5.4) \quad \begin{aligned} \ln(Z_T) = \ln(Z'_T) + \ln(Z''_T) = & - \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} S_t^{-1} dS_t + \int_0^T \left( \frac{H_t(t, Y_t)}{H(t, Y_t)} - \lambda Y_t \frac{H_y(t, Y_t)}{H(t, Y_t)} \right) dt \\ & + \int_0^T \int_0^\infty (\ln H(t, Y_t + z) - \ln H(t, Y_t)) N(dz, dt). \end{aligned}$$

Since  $H \in C^{1,1}$  from Prop. 3.1, we can apply Itô's formula on  $g(t, Y_t) = \ln H(t, Y_t)$  to derive

$$(5.5) \quad \begin{aligned} g(T, Y_T) = & g(0, Y_0) + \int_0^T \left( \frac{H_t(t, Y_t)}{H(t, Y_t)} - \lambda Y_t \frac{H_y(t, Y_t)}{H(t, Y_t)} \right) dt \\ & + \int_0^T \int_0^\infty (\ln H(t, Y_t + z) - \ln H(t, Y_t)) N(dz, dt). \end{aligned}$$

Finally, substitution of (5.5) in (5.4) yields

$$(5.6) \quad \begin{aligned} Z_T = & \exp \left( -\ln H(0, y) - \int_0^T \frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1} dS_t \right) \\ & \exp \left( -\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t - \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right) \\ = & \frac{\exp \left( -\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t - \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right)}{\mathbb{E} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right]}, \end{aligned}$$

such that  $\eta_t$  is given by  $-\frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1}$ .

**ii)** By assumption (5.1) and the Novikov condition, we know that  $Z'_t$  is a true martingale. We denote its corresponding probability measure by  $Q'$  and remind that  $Y_t$  has the same dynamics under  $P$  and  $Q'$ . So we get

$$(5.7) \quad \mathbb{E}[Z_T] = \frac{\mathbb{E} \left[ Z'_T \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right]}{\mathbb{E} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right]} = \frac{\mathbb{E}_{Q'} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right]}{\mathbb{E} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right]} = 1.$$

This shows that  $Z_t$  is a martingale.

**iii)** Using the same arguments as in ii), we see that

$$(5.8) \quad \begin{aligned} \mathbb{E}[Z_T | \ln Z_T] = & \mathbb{E}_{Q'} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \mid \left( \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t + \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right) \mid \right] \\ = & \mathbb{E}_{Q'} \left[ \exp \left( -\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \mid \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} d\tilde{B}_t \right] < \infty, \end{aligned}$$

where  $\tilde{B}_t$  is the Brownian motion under  $Q'$ .

iv) Since we have

$$\exp \left( \int_0^T \eta_t^2 d[S]_t \right) = \exp \left( \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right),$$

we get by assumption (5.1) that  $\int_0^T \eta_t^2 d[S]_t \in L_{exp}(P)$ .  $\square$

Theorem 5.1 proves that under assumption (5.1) the candidate derived in Section 3 and 4 by applying the dynamic programming method actually is the density process of the MEMM. The remaining task is to provide sufficient conditions such that assumption (5.1) is fulfilled in our model. The following Proposition gives a sufficient condition depending on the Lévy measure of  $L_1$ , and determines an exact constant  $c$  in the exponential integrability condition (2.8):

**Proposition 5.2.** *If*

$$(5.9) \quad \int_0^\infty \left\{ \exp \left( \frac{\beta^2}{\lambda} (1 - \exp(-\lambda T)) z \right) - 1 \right\} \nu(dz) < \infty,$$

*then  $Z_T$  is the density process of the minimal entropy martingale measure.*

*Proof.* We have

$$\frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} = \frac{\mu^2}{Y_t} + 2\mu\beta + \beta^2 Y_t \leq C + \beta^2 Y_t$$

for a constant  $C$ . This is because  $Y_t \geq y \exp(-\lambda T)$ . But this gives

$$(5.10) \quad \begin{aligned} \mathbb{E} \left[ \exp \left( \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right) \right] &\leq C' \mathbb{E} \left[ \exp \left( \beta^2 \int_0^T Y_t dt \right) \right] \\ &= C' \mathbb{E} \left[ \exp \left( \frac{\beta^2}{\lambda} \left( y(1 - \exp(-\lambda T)) + \int_0^t (1 - \exp(-\lambda(T-t))) dL_{\lambda t} \right) \right) \right] \\ &= C'' \exp \left( \lambda \int_0^T \int_0^\infty (\exp(f(t)z) - 1) \nu(dz) dt \right), \end{aligned}$$

where  $C'$ ,  $C''$  are constants and  $f(t) = \frac{\beta^2}{\lambda} (1 - \exp(-\lambda(T-t)))$ .  $\square$

With the verification of the candidate density, we have identified the minimal entropy martingale measure. An implication of this is that we rigorously can state from arbitrage theory that the Feynman-Kac representation of  $\Lambda(t, y, s)$  in (4.2) is the minimal entropy price of the claim  $f(S_T)$ , and moreover, that this pricing function must solve the integro-type Black & Scholes equation (4.1).

We consider some examples of the process  $L_t$  that are relevant in finance, and state sufficient conditions for the density of the minimal entropy martingale measure. If we choose the stationary distribution of  $Y_t$  to be an inverse Gaussian law with parameters  $\delta$  and  $\gamma$ , that is  $Y_y \sim \text{IG}(\delta, \gamma)$ , the Lévy measure of  $L$  becomes

$$\nu(dz) = \frac{\delta}{2\sqrt{2\pi}} z^{-3/2} (1 + \gamma z) \exp \left( -\frac{1}{2} \gamma z \right) dz.$$

Hence, the exponential integrability condition in Prop. 5.2 is satisfied whenever

$$\beta^2 (1 - \exp(-\lambda T)) < \frac{1}{2} \gamma^2 \lambda.$$

When  $Y_t \sim IG(\delta, \gamma)$ , the logreturns of  $S_t$  will be approximately normal inverse Gaussian distributed, a family of laws that has been successfully fitted to logreturns of stock prices (see e.g., Barndorff-Nielsen [2] and Rydberg [18]).

Another popular distribution in finance is the variance gamma law (see Madan and Seneta [13]). If the stationary distribution of  $Y_t$  is a gamma law with parameters  $\delta$  and  $\alpha$ , that is  $Y_t \sim \Gamma(\delta, \alpha)$ , the marginal distribution of the logreturns of  $S_t$  is approximately a variance gamma law. The Lévy measure of  $L$  becomes

$$\nu(dz) = \delta \alpha \exp(-\alpha z) dz,$$

for which the integrability condition in Prop. 5.2 is satisfied whenever

$$\beta^2 (1 - \exp(-\lambda T)) < \alpha \lambda.$$

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